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Darboux transformations of coherent states of the time-dependent singular oscillator

Boris F Samsonov

Physics Department of Tomsk State University, 36 Lenin Ave, 634050 Tomsk, Russia
and
Departamento de Física Teórica, Universidad de Valladolid, 47005 Valladolid, Spain

E-mail: samsonov@phys.tsu.ru

Received 16 January 2004, in final form 9 June 2004

Published 14 October 2004

Online at stacks.iop.org/JPhysA/37/10273

doi:10.1088/0305-4470/37/43/017

Abstract

Darboux transformation of both Barut–Girardelo and Perelomov coherent states for the time-dependent singular oscillator is studied. In either case, a measure that realizes the resolution of the identity operator in terms of coherent states is found and a corresponding holomorphic representation is constructed. For the particular case of the free particle moving with the fixed value of the square of angular momentum equal to 2 it is shown that the Barut–Girardelo coherent states are more localized at the initial time moment while the Perelomov coherent states are more stable with respect to time evolution. It is also illustrated that Darboux transformation may keep unchanged this different time behaviour.

PACS numbers: 03.65.Fd, 11.30.Pb

1. Introduction

There exist several definitions of coherent states (CS) [1–3]. Different definitions lead to the same result for the harmonic oscillator potential and usually different results for other physical systems. Nevertheless, a careful analysis shows that different CS possess common properties which can be considered to define CS for the very general quantum system [3]. It happens that such a definition is very ambiguous. Therefore, one can choose from different possible systems the one which has a desirable property. For instance, one can look for a state the wavefunction of which can be expressed in terms of known special functions or even in terms of elementary functions only. This simplifies considerably the study of the CS and their use in other applications. In this way, one could construct different systems of CS for the well-known soliton potentials both for time-dependent and time-independent cases [4].

The most essential progress in studying CS has been achieved for systems possessing symmetries [2]. If it is not possible to associate a symmetry group with a quantum system the problem becomes much more complicated. For instance, after applying a nontrivial supersymmetry (or equivalently Darboux) transformation to a Hamiltonian allowing for a symmetry, the symmetry is usually lost or it is transformed to a nonlinear symmetry [5]. This makes it impossible to apply group theoretical methods for studying CS. In this respect, it was suggested applying the same transformation to the known CS of the initial quantum system and treating states thus obtained as coherent states for the transformed system [6, 7]. Such a possibility is based on the fact that these states fulfil all properties of coherent states formulated by Gazeau and Klauder [3]. This approach has proved to be useful for studying CS for supersymmetrically transformed harmonic oscillator [7] and time-independent singular oscillator [8] potentials. In particular, using correspondence between classical and quantum systems, which can be realized just by the technique of CS (see, e.g., [2]), one could give an interpretation of the supersymmetry transformation in terms of classical notions [5] as a transformation of the Kähler potential. Here, we continue this study at the level of the time-dependent singular oscillator.

The time-dependent singular oscillator Hamiltonian

$$h_0 = -\partial_x^2 + \omega^2(t)x^2 + gx^{-2} \quad (1)$$

plays an essential role in different physical applications, among which we would like to mention interesting results in molecular physics [9], optics [10] and mathematical physics [11]. The concept of even and odd coherent states [12] based on generalized coherent states of the Hamiltonian (1) serves as the simplest model of a quantum microscopical superposition, the problem raised by Schrödinger [13] and known as Schrödinger cat states. They appear as photon nonclassical states in cavity experiments [14], in studying optical properties of a nonlinear Kerr medium [15] and in experiments with trapped ions [16]. To obtain a qualitative description of the phenomenon and more or less good numerical agreement between theoretical results and experimental data it was necessary to introduce in the theory various nonlinear modifications [17]. Recently, a model of a two-ion trap has been proposed based on this Hamiltonian (see [18] where a good literature review is also given).

From this point of view, the CS we are studying in this paper give an example of highly nonlinear coherent states which take into consideration a certain perturbation of the simple interaction presented in the Hamiltonian (1). This perturbation is not arbitrary. It is consistent with the fact that the perturbed Hamiltonian h_1 is related to h_0 by a time-dependent Darboux transformation. For the Hamiltonian h_0 two systems of CS are commonly known, these are Barut–Girardelo CS and Perelomov CS (for the definitions, see section 2). We apply the same Darboux transformation operator to both systems of CS thus obtaining two different systems of solutions of the Schrödinger equation with the Hamiltonian h_1 . They satisfy all the properties formulated by Gazeau and Klauder [3] for the states to be CS except maybe for the resolution of the identity operator. Next, for both systems of the above states we find a measure that realizes the resolution of the identity operator in terms of these states thus proving the main result of the paper that in both cases we obtain coherent states in the sense of the definition by Gazeau and Klauder. Different systems of CS for the Hamiltonian h_0 have different properties. In particular, the Barut–Girardelo CS may be more localized at the initial time moment while the Perelomov CS are more stable in time. For a particular example of the free particle moving in a central field with the angular quantum number $\ell = 1$ we show that this different time behaviour of different CS is practically unaffected by the Darboux transformation and conjecture that this is a reflection of a more general statement that other properties of CS may be approximately kept after the Darboux transformation is realized.

Our results permit us to hope that the new systems of CS may be useful in describing similar experiments where the initial systems have proved to be useful with the advantage that an additional time-dependent member may be included in the Hamiltonian to approximate a real interaction better.

Another case that we would like to mention here is related to the possibility of obtaining time-dependent spherically symmetric exactly solvable three-dimensional interactions when the parameter g in (1) is related to the orbital quantum number ℓ , $g = \ell(\ell + 1)$. An example will be discussed in section 3.

2. Singular oscillator Hamiltonian

In this section, we summarize briefly what is known about this Hamiltonian and its coherent states (see, e.g., [5, 8, 18–20]).

In contrast to most papers using different modifications of the method of quantum invariants [21] for solving the Schrödinger equation, we are using the method of separation of variables [22]. The advantage of this approach is that we do not need to be restricted by any quantum mechanical picture. Instead, we consider the Schrödinger equation as a second-order parabolic differential equation, which gives us the possibility of getting the ‘nonphysical’ solutions we need for applying the Darboux algorithm.

2.1. Solutions of the Schrödinger equation

Generators of the $SU(1, 1)$ symmetry group, being symmetry operators of the Schrödinger equation with the Hamiltonian h_0 , in the coordinate representation

$$k_- = 2[a^2 - \varepsilon^2 g x^{-2}] \quad k_+ = 2[(a^+)^2 - \bar{\varepsilon}^2 g x^{-2}] \quad (2)$$

$$k_0 = \frac{1}{2}(k_- k_+ - k_+ k_-) \quad (3)$$

are expressed in terms of the harmonic oscillator creation and annihilation operators

$$a = \varepsilon \partial_x - \frac{i}{2} \dot{\varepsilon} x \quad a^+ = -\bar{\varepsilon} \partial_x + \frac{i}{2} \dot{\bar{\varepsilon}} x.$$

The dot over a symbol means the derivative with respect to time. Parameters ε and $\bar{\varepsilon}$ are linearly independent solutions to the equation of motion for the classical harmonic oscillator

$$\ddot{\varepsilon} + 4\omega^2(t)\varepsilon = 0.$$

In particular, when ε is complex, $\bar{\varepsilon}$ will denote its complex conjugate and they are such that $\dot{\varepsilon}\bar{\varepsilon} - \varepsilon\dot{\bar{\varepsilon}} = \frac{1}{2}i$. Casimir operator C is expressed in terms of the parameter g : $C = \frac{1}{2}(k_+ k_- + k_- k_+) - k_0^2 = \frac{3}{16} - \frac{1}{4}g$. This gives the relation between g and the representation parameter k : $C = k(1 - k)$, $g = \frac{3}{4} + 4k(k - 1)$. We would like to mention that by definition of a symmetry operator of a linear differential equation (see, e.g., [22]) the operators (2), (3) commute with the Schrödinger operator $i\partial_t - h_0$ with h_0 as given in (1) on the space of solutions of the Schrödinger equation and therefore they are integrals of motion for this equation. The Hamiltonian (1) is not an integral of motion (which reflects the fact that the energy is not a conserving quantity for a time-dependent potential) and cannot be expressed as a linear combination of the operators (2), (3) with time-independent coefficients.

Square integrable solutions $\psi_n(x, t)$, $n = 0, 1, \dots$, of the Schrödinger equation satisfying the zero boundary condition at the origin form a basis of the discrete series irreducible representation $T_k^+(g)$ of the group $SU(1, 1)$ at $k = \frac{1}{2} + \frac{1}{4}\sqrt{1 + 4g}$ in the Hilbert space $L_2(0, \infty)$. They can easily be obtained starting from the vacuum state (see, e.g., [2]). But to be able

to apply below the time-dependent Darboux algorithm we need solutions outside the Hilbert space $L_2(0, \infty)$. To get them we proceed to solving the Schrödinger equation with the Hamiltonian (1).

To solve the Schrödinger equation simultaneously with the eigenvalue equation for k_0 we are using the method of separation of variables. From coordinates (x, t) we are going to curvilinear coordinates (ξ, τ) defined as $\xi = x\gamma^{-1/2}$, $\tau = t$ where $\gamma = \varepsilon\bar{\varepsilon}$. The choice

$$\psi = e^{\frac{i}{8}\xi^2\dot{\gamma}} P(\xi)Q(\tau) \quad (4)$$

guarantees the separation of variables. The eigenvalue equation for k_0 , $k_0\psi = \lambda\psi$, is reduced to the first-order ordinary equation for P which can readily be integrated to give $P = \gamma^{-1/4}(\bar{\varepsilon}/\varepsilon)^\lambda$. So, replacing the action of k_0 on a solution by multiplication on the separation constant λ one obtains from the Schrödinger equation the following second-order differential equation:

$$Q''(y) - \left[\frac{1}{4}y^2 + gy^{-2} - 2\lambda\right] Q(y) = 0 \quad y = \frac{1}{2}\xi. \quad (5)$$

This equation can be reduced to the equation for the Laguerre polynomials if $\lambda = n + k$, $n = 0, 1, \dots$, which just corresponds to the well-known coordinate representation of the basis functions $\psi_n(x, t)$. If $\lambda = -k - n$ or $\lambda = k - n - 1$, $n = 0, 1, \dots$, it can give rise to Laguerre polynomials also, but of course this does not correspond to square integrable solutions of the Schrödinger equation with the zero boundary condition at the origin. Finally, all these solutions have the form

$$\psi_n = N_{0n}\gamma^{-1/4} \left(\frac{\bar{\varepsilon}}{\varepsilon}\right)^\lambda y^{\alpha+\frac{1}{2}} \exp\left(\left(\pm\frac{1}{4} + \frac{i}{2}\dot{\gamma}\right)y^2\right) L_n^\alpha\left(\mp\frac{1}{2}y^2\right) \quad y = \frac{1}{2}\gamma^{-\frac{1}{2}}x. \quad (6)$$

Here, α is defined by the equation $\alpha^2 = (2k - 1)^2$. The lower sign in (6) and the choice $\alpha = 2k - 1$ correspond to the discrete spectrum eigenfunctions. The upper sign permitting α to be both positive and negative corresponds to solutions which do not belong to the Hilbert space $L_2(0, \infty)$. For $\alpha = 2k - 1$ they are such that $k_0\psi_n(x, t) = -(k + n)\psi_n(x, t)$ and for $\alpha = 1 - 2k$ they satisfy the equation $k_0\psi_n(x, t) = (k - n - 1)\psi_n(x, t)$. The value $N_{0n} = 2^{3k-\frac{1}{2}}(n!)^{\frac{1}{2}}\Gamma^{-\frac{1}{2}}(n+2k)$ guarantees the normalization of the square integrable solutions to the unity. For the non-normalizable solutions N_{0n} does not play any role. It is easily verified that the action of the operators (2) and (3) on the functions (6) when they form a basis in the Hilbert space $L_2(0, \infty)$ is given by

$$\begin{aligned} k_0\psi_n(x, t) &= (k + n)\psi_n(x, t) & k_\pm\psi_n(x, t) &= -c_n^\pm\psi_{n\pm 1}(x, t) \\ c_n^\pm &= \left(n + \frac{1}{2} \pm \frac{1}{2}\right)^{\frac{1}{2}} \left(n + 2k - \frac{1}{2} \pm \frac{1}{2}\right)^{\frac{1}{2}}. \end{aligned} \quad (7)$$

For applying the Darboux algorithm below we need nodeless solutions. The zeros of the Laguerre polynomials are well known [23]. In the negative semiaxis, $x < 0$, $L_n^\alpha(x)$ has only one node provided $(\alpha + 1)_n = (\alpha + 1)(\alpha + 2) \cdots (\alpha + n) < 0$. It is clear that since we put $k = \frac{1}{2} + \frac{1}{4}\sqrt{1+4g}$, $\alpha = 2k - 1 = \frac{1}{2}\sqrt{1+4g} > 0$ meaning that in this case any ψ_n is nodeless and it is such that $\psi_n^{-1}(\infty) = 0$ but $\psi_n^{-1}(0) = \infty$. For a negative α the result depends on the interval where α falls. If $-2m - 1 > \alpha > -2m - 2$, we have $n = 0, 2, \dots, 2m$ and for $-2m \geq \alpha \geq -2m - 1$ more values of n are possible: $n = 0, 2, \dots, 2m, 2m + 1, 2m + 2, \dots$, where $m = 0, 1, \dots$. Moreover, for $\alpha < -3/2$ the function $\psi_n^{-1}(x)$ is square integrable and belongs to the domain of definition of h_0 when it is considered as an operator in the Hilbert space $L_2(0, \infty)$.

2.2. Barut–Girardelo coherent states

The states $\psi_\mu(x, t)$ known as Barut–Girardelo CS [20] are defined as the eigenstates of the operator k_-

$$k_- \psi_\mu(x, t) = \mu \psi_\mu(x, t) \quad \mu \in \mathbb{C}. \tag{8}$$

To solve this equation simultaneously with the Schrödinger equation, we are using the method of separation of variables once again. In coordinates $\{\xi = x/\varepsilon, \tau = t\}$, the Schrödinger equation separates if $\psi_\mu = \exp(\frac{1}{4}\varepsilon\dot{\xi}\xi^2)P(\xi)Q(\tau)$. The function $P(\xi)$ is defined by a first-order ordinary equation which is easily integrated to give $P = \varepsilon^{-\frac{1}{2}} e^{-2\mu^2\frac{\xi}{\varepsilon}}$. The Schrödinger equation is reduced to the following second-order equation for Q :

$$Q''(\xi) - (g\xi^{-2} + \mu^2)Q(\xi) = 0 \tag{9}$$

which after changing the dependent variable $Q = \xi^{\frac{1}{2}}I$ becomes the modified Bessel equation for I . The condition for $\psi_\mu(x, t)$ belong to the domain of definition of h_0 , when it is considered as an operator acting in the space $L_2(0, \infty)$, selects for us only one solution to this equation: $I = I_{2k-1}(\mu\xi)$. (We are using the standard notation for the Bessel functions [23].) To calculate the normalization integral, we are making use of tables [24]. So, the normalized solution is

$$\psi_\mu(x, t) = \frac{1}{2\varepsilon} I_{2k-1}^{-\frac{1}{2}}(4\mu\bar{\mu})\sqrt{x} I_{2k-1}\left(\frac{1}{\varepsilon}\mu x\right) \exp\left(\frac{i\dot{\xi}}{4\varepsilon}x^2 - \frac{2\bar{\varepsilon}}{\varepsilon}\mu^2\right). \tag{10}$$

Expanding the Bessel function here in the Taylor series one gets their expansion in terms of Laguerre polynomials and in such a way they are expressed in terms of the basis functions (6) with the coefficients depending only on the even powers of μ . Therefore, it is more convenient to change the complex variable μ in favour of $\lambda = 2\mu^2$ which yields

$$\psi_\lambda(x, t) = N_{0\lambda} \sum_{n=0}^{\infty} a_n \lambda^n \psi_n(x, t) \quad a_n = \frac{(-1)^n \sqrt{\Gamma(2k)}}{\sqrt{n! \Gamma(n+2k)}} \quad \lambda \in \mathbb{C} \tag{11}$$

with

$$N_{0\lambda} = (\bar{\lambda}\lambda)^{\frac{k}{2}-\frac{1}{4}} \Gamma^{-\frac{1}{2}}(2k) I_{2k-1}^{-\frac{1}{2}}(2|\lambda|). \tag{12}$$

As usual, the states $\psi_\lambda(x, t)$ are not orthogonal to each other

$$\langle \psi_{\lambda'} | \psi_\lambda \rangle = \frac{I_{2k-1}(2\sqrt{\lambda\bar{\lambda}'})}{[I_{2k-1}(2|\lambda|)I_{2k-1}(2|\lambda'|)]^{\frac{1}{2}}}.$$

Since the basis $\{\psi_n(x, t)\}$ is complete in the Hilbert space one can calculate [8] the measure $\rho(\lambda)$ which realizes the resolution of the identity over the coherent states ψ_λ

$$\int |\psi_\lambda\rangle \langle \psi_\lambda| d\rho(\lambda) = 1 \quad \rho(\lambda) = \frac{1}{\pi} K_{2k-1}(2|\lambda|) I_{2k-1}(2|\lambda|) d\lambda d\bar{\lambda}. \tag{13}$$

All integrals over the variable λ are extended to the whole complex plan.

Now one can construct a holomorphic representation of the vectors and operators [8]. Any $\psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x, t) \in L_2(0, \infty)$ can be written in the coherent state representation $\psi^c(\lambda)$:

$$\psi^c(\lambda) := \langle \psi_{\bar{\lambda}}(x, t) | \psi(x, t) \rangle = N_{0\lambda} \sum_{n=0}^{\infty} a_n c_n \lambda^n \equiv N_{0\lambda} \psi(\lambda) \quad \lambda \in \mathbb{C}. \tag{14}$$

The holomorphic function $\psi(\lambda) = \sum_{n=0}^{\infty} a_n c_n \lambda^n$ can be associated with any function $\psi(x, t) \in L_2(0, \infty)$ given by its Fourier coefficients c_n over the basis (6).

Using the complex conjugate form of the resolution of the identity (13) one can define an inner product $\langle \psi_a(\lambda) | \psi_b(\lambda) \rangle$ in the space of the functions $\psi_{a,b}(\lambda)$ holomorphic in the complex plane

$$\langle \psi_a | \psi_b \rangle = \int |N_{0\lambda}|^2 \bar{\psi}_a(\lambda) \psi_b(\lambda) d\rho(\lambda) := \langle \psi_a(\lambda) | \psi_b(\lambda) \rangle. \quad (15)$$

This means that the integration in the space of holomorphic functions should be carried out with the measure $d\tilde{\rho}(\lambda) = |N_{0\lambda}|^2 d\rho(\lambda)$, so that

$$\langle \psi_a(\lambda) | \psi_b(\lambda) \rangle = \int \bar{\psi}_a(\lambda) \psi_b(\lambda) d\tilde{\rho}(\lambda). \quad (16)$$

To distinguish this inner product from the one in the space $L^2(0, \infty)$ we indicate the integration variable λ inside the parentheses. The space of holomorphic functions $\psi(\lambda)$ such that

$$\int |\psi(\lambda)|^2 d\tilde{\rho}(\lambda) < \infty$$

equipped with the inner product (16) becomes the Hilbert space.

The orthonormal basis $\psi_n(x, t)$ in this representation looks as follows:

$$\psi_n(\lambda) = a_n \lambda^n \quad \langle \psi_n(\lambda) | \psi_{n'}(\lambda) \rangle = \delta_{nn'}.$$

The holomorphic representation of the Dirac-delta function, coherent states and the $SU(1, 1)$ generators are exactly the same as for the time-independent case [8]. The mean value of the operator k_0 in the coherent state

$$\langle \psi_\lambda | k_0 | \psi_\lambda \rangle = k + |\lambda| \frac{I_{2k+1}(2|\lambda|)}{I_{2k-1}(2|\lambda|)} \quad (17)$$

may be useful in the following.

2.3. Perelomov coherent states

Perelomov CS $\psi_z(x, t)$ are obtained by acting on the ground-state function with the group translation operator (see, e.g., [2]). Therefore, their Fourier coefficients over the basis $\psi_n(x, t)$ are independent of t and coincide with the ones for the time-independent frequency ω

$$\psi_z(x, t) = N_{0z} \sum_{n=0}^{\infty} a_n z^n \psi_n(x, t) \quad |z| < 1 \quad (18)$$

$$N_{0z} = (1 - |z|^2)^k \quad a_n = \sqrt{\frac{\Gamma(n+2k)}{n! \Gamma(2k)}}. \quad (19)$$

Using the generating function for the Laguerre polynomials one gets their explicit expression

$$\psi_z(x, t) = 2^{\frac{1}{2}-3k} \Gamma^{-\frac{1}{2}}(2k) e^{-2k} x^{2k-\frac{1}{2}} \left(\frac{1-|\zeta|^2}{(1-\zeta)^2} \right)^k \exp\left(-\frac{x^2}{16\gamma} \frac{1+\zeta}{1-\zeta} + \frac{ix^2\dot{\gamma}}{8\gamma} \right) \quad (20)$$

where $\zeta = z \frac{\dot{\gamma}}{\gamma}$ and $|z| < 1$.

We note that since the series expansion (18) of the time-dependent CS (20) in terms of the time-dependent basis $\{\psi_n(x, t)\}$ is exactly the same as the expansion for the time-independent Hamiltonian, all properties of such states known for the stationary case take place for the non-stationary one. In particular, they realize the resolution of the identity operator and, hence, one can map any square integrable on the positive semiaxis function to a function holomorphic in the unit disc getting in such a way a holomorphic representation. We will not go in details for this case since they are very well known [2].

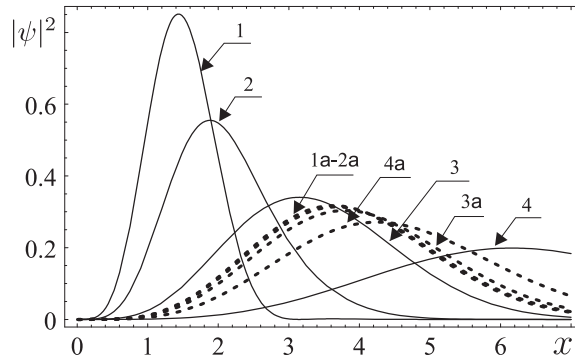


Figure 1. Comparison between probability distribution for Barut–Girardelo and Perelomov CS for time moments $t = 0$: 1 and 1a; $t = 0.5$: 2 and 2a; $t = 1$: 3 and 3a; $t = 2$: 4 and 4a.

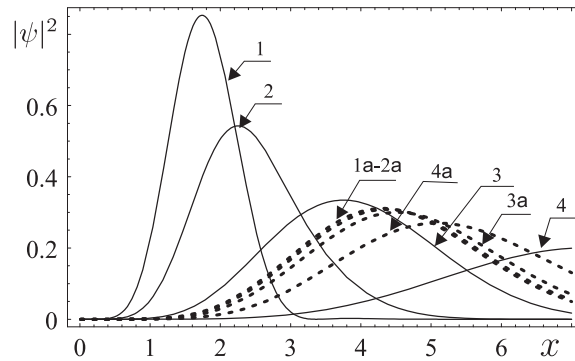


Figure 2. Comparison between probability distribution for Darboux transformed Barut–Girardelo and Perelomov CS for time moments $t = 0$: 1 and 1a; $t = 0.5$: 2 and 2a; $t = 1$: 3 and 3a; $t = 2$: 4 and 4a.

2.4. Discussion

Since the parameters z and λ labelling the CS fill different domains, it is not straightforward to make a comparison between them. One of the possibilities could be to choose these parameters such that the mean value of an operator is the same for both states. As a particular example, we take $\omega = 0$ (free particle) and $g = 2$ (orbital quantum number $\ell = 1$). In this case, $\varepsilon = \frac{1}{\sqrt{2}}(t + i)$ and we have chosen real values for z and λ such that the mean value of k_0 coincides with its value in the first excited state (6) which is $k + 1$. In this case, $z = (2k + 1)^{-1/2}$ and $\lambda \simeq 1.021$. With these values of z and λ , we plotted the squared modulus of $\psi_z(x, t)$ (20) and $\psi_\lambda(x, t)$ given by (10) at $\mu = \sqrt{\lambda}/2$ in figure 1. It is clearly seen from the figure that at the initial time moment the Barut–Girardelo CS are much more localized than the Perelomov CS but the latter are much more stable in time. At times greater than 2, they are already very spread whereas the Perelomov CS maintain almost the same localization as at the initial time moment. In the next section, we show that the Darboux transformation keeps the different time behaviour of the two types of states practically unchanged.

3. Darboux transformation of coherent states

There are several options to realize Darboux transformations for the time-dependent Schrödinger equation. The method based on differential transformation operators developed by Matveev and Salle [25] gives in general complex potential differences since it uses the second logarithmic derivative with respect to the spatial variable of a given solution of the Schrödinger equation which is a complex valued function. For instance, for the free particle one can choose proper solutions to get the well-known soliton potentials for the Kadomtsev–Petviashvili I equation, but for the Hamiltonian (1) the solutions of the form (6) produce only complex potential differences. This inconvenience is not present in the binary transformation [25] (see also [26] and for recent developments [27]), but the transformation operator now has an integral form which usually is less convenient in practical calculation than the differential one. In [28], the method of differential transformation operators is modified to get real exactly solvable potentials which we find the most convenient for our purpose. According to this approach, real nontrivial potentials are possible if the Schrödinger equation has at least one solution $u = u(x, t)$ satisfying the *reality condition* $(\log u/\bar{u})_{xxx} = 0$. The solutions (4) satisfy this condition for any real function Q . So, taking different real solutions of equation (9) one can, in general, obtain a two-parameter family of exactly solvable partners for h_0 , but only the functions $u = \psi_n(x, t)$ (6) produce potentials expressed in terms of elementary functions. Therefore in what follows, we restrict ourselves by considering this particular choice of transformation functions only. Nevertheless, it is useful to note that the use of N iterated transformations can give a $2N$ -parameter family of exactly solvable potentials which in our case becomes N parameter and the parameters take only discrete values. Since the iteration procedure is a straightforward generalization of the simple one-step transformation, we will illustrate our ideas using only the simplest one-step transformations.

Between the square integrable solutions only $\psi_0(x, t)$ is nodeless. Unfortunately, it produces a new potential of the same kind as V_0 changing the value of g only (shape invariance at time-dependent level). So, to get nontrivial new potentials, we have to keep in (6) only the upper sign. For $\alpha < -3/2$ any nodeless function (6) gives rise to a new potential. The values of the parameter α for which it is nodeless are indicated at the end of section 2.1. For simplicity, we will consider here only $\alpha > 0$ when all square integrable and vanishing at the origin solutions for the transformed Hamiltonian $h_1 = -\partial_x^2 + V_1(x, t)$ can be obtained by acting on corresponding solutions of the initial equation with the transformation operator

$$L = L_1(t)[\partial_x - u_x(x, t)/u(x, t)] \quad L_1(t) = \exp\left[2 \int dt \operatorname{Im}(\ln u)_{xx}\right]. \quad (21)$$

The potential $V_1(x, t)$ is expressed in terms of the same function u :

$$V_1(x, t) = V_0(x, t) + A(x, t) \quad A(x, t) = -[\ln |u(x, t)|^2]_{xx}.$$

Taking one of the functions (6) with $n = m$ one gets a one-parameter (labelled by m) family of the potential differences $A(x, t) = A_m(x, t)$

$$A_m(x, t) = \frac{4k-1}{x^2} + \frac{1}{8} \left(\frac{xL_{m-1}^{2k}(z)}{\gamma L_m^{2k-1}(z)} \right)^2 - \frac{x^2 L_{m-2}^{2k+1}(z) + 4\gamma L_{m-1}^{2k}(z)}{8\gamma^2 L_m^{2k-1}(z)} - \frac{1}{4\gamma} \quad z = -\frac{x^2}{8\gamma}. \quad (22)$$

They are expressed in terms of Laguerre polynomials and therefore for any integer m only elementary functions are involved in this expression. Here and in the following, the Laguerre

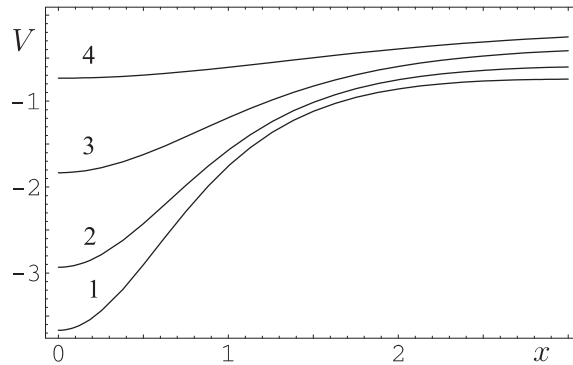


Figure 3. Potentials (25) at $\ell = 0$ and at different time moments: 1: $t = 0$; 2, $t = 0.5$; 3: $t = 1$; 4, $t = 2$.

polynomials with a negative value of the subscript should be put equal to zero. For instance, for $\omega(t) = 0$ and $m = 1$ the new potential reads

$$V_1(x, t) = \frac{(\ell + 1)(\ell + 2)}{x^2} + \tilde{V}_1(x, t) \tag{23}$$

where

$$\tilde{V}_1(x, t) = -\frac{1}{1+t^2} - \frac{4}{x^2 + (2\ell + 3)(1+t^2)} + \frac{8x^2}{(x^2 + (2\ell + 3)(1+t^2))^2} \tag{24}$$

and we put $g = \ell(\ell + 1)$. For $m = 2$, the expression for $\tilde{V}_1(x, t)$ is still reasonable

$$\tilde{V}_1(x, t) = -\frac{1}{1+t^2} - \frac{8}{X}(3x^2 + (5 + 2\ell)(1+t^2)) + \frac{32}{X^2}((x^3 + (5 + 2\ell)(1+t^2)x)^2) \tag{25}$$

where $X = x^4 + 2(5 + 2\ell)(1+t^2)x^2 + (3 + 2\ell)(5 + 2\ell)(1+t^2)^2$. It is also clear that if we consider x in (24) and (25) as the radial variable, $x = r$, we can solve the three-dimensional Schrödinger equations with these potentials since the angular variables being separated in this equation with the help of the spherical harmonic $Y_{\ell+1,m}(\theta, \varphi)$ and the corresponding point transformation of the wavefunction being made one gets from the three-dimensional Schrödinger equation the one-dimensional equation with the potential (23), for which we know how to get solutions. The potentials (24) and (25) look like potential wells with the depth decreasing in time for $t \geq 0$ and they are even functions of time. Their typical behaviour is shown in figure 3. We would also like to note that for a periodical time dependence of $\omega(t)$, it is possible to manage quasienergy levels and compute the change of the geometric phase produced by the Darboux transformation in a way similar to that already reported in [29] for the time-dependent harmonic oscillator. This work is currently in progress.

The function $L_1(t)$ is determined by (21) up to a multiplicative constant which we fix to simplify subsequent formulae so that $L_1 = \sqrt{2\gamma}$ and the explicit expression for the transformation operator (21) is

$$L = \sqrt{2\gamma} \left[\partial_x - \frac{x}{8\gamma} - \frac{4k-1}{2x} - \frac{ix\dot{\gamma}}{4\gamma} - \frac{xL_{m-1}^{2k}}{4\gamma L_m^{2k-1}} \right]. \tag{26}$$

Normalized to unity, solutions for the Hamiltonian h_1 are $\varphi_n = N_{1n}L\psi_n$, $N_{1n} = (n+2k+m)^{-\frac{1}{2}}$.

Operator L and its formally adjoint L^+ factorize the symmetry operator k_0 : $L^+L = k_0 + k + m$. The opposite superposition LL^+ is a symmetry operator for the transformed

Schrödinger equation. If we denote $p_0 = LL^+ - k - m$ then the functions φ_n are eigenfunctions of p_0 and the spectrum of p_0 is identical to the spectrum of k_0 . Using k_{\pm} and transformation operators L, L^+ one can construct other symmetry operators for the transformed equation: $p_{\pm} = L\kappa_{\pm}L^+$. They are ladder operators for the functions φ_n :

$$p_{\pm}\varphi_n(x, t) = (N_{1n}N_{1(n\pm 1)})^{-1}c_n^{\pm}\varphi_{n\pm 1}(x, t).$$

We would like to note that the Hamiltonian h_1 gives us an example of the Schrödinger equation, symmetry operators of which do not close a Lie algebra but satisfy a polynomial algebra

$$[p_0, p_{\pm}] = \pm p_{\pm}$$

$$[p_-, p_+] = 2[k(1-k) + 2p_0(k+m) + 2p_0^2](p_0 + k + m).$$

Similar algebra was previously obtained for the time-independent singular oscillator in [5].

3.1. Transformation of Barut–Girardello coherent states

To obtain CS for the Hamiltonian h_1 we act with L given in (26) on CS ψ_{λ} : $\varphi_{\lambda} = N_{1\lambda}L\psi_{\lambda}$. The factor $N_{1\lambda}$ being calculated from the formula $N_{1\lambda}^{-2} = \langle \psi_{\lambda} | k_0 + k + m | \psi_{\lambda} \rangle$ and (17) guarantees the normalization of the states φ_{λ} to unity. Their series expansion in terms of the basis $\{\varphi_n\}$ can be found by acting with the same operator on the series (11):

$$\varphi_{\lambda} = N \sum_{n=0}^{\infty} b_n \lambda^n \varphi_n \quad b_n = a_n (n + 2k + m)^{\frac{1}{2}} (2k + m)^{-\frac{1}{2}} \quad \lambda \in \mathbb{C} \quad (27)$$

where $N = (2k + m)^{\frac{1}{2}} N_{1\lambda} N_{0\lambda}$.

The states $\varphi_{\lambda} = \varphi_{\lambda}(x, t)$ thus obtained may be interpreted as coherent states if they admit the resolution of the identity operator

$$\int |\varphi_{\lambda}\rangle \langle \varphi_{\lambda}| d\tilde{\rho}_1(\lambda) = 1. \quad (28)$$

Now we proceed to find the measure $\tilde{\rho}_1(\lambda)$. We will look for the function $\tilde{\rho}_1(\lambda)$ depending only on the absolute value of λ , $|\lambda| = \sqrt{x}$: $d\tilde{\rho}_1 = \frac{1}{2}h(x) dx d\phi$; the function $h(x)$ is to be determined.

It is convenient to use the polar coordinates in the complex plane of the variable λ , $\lambda = \sqrt{x} \exp(i\phi)$. After being integrated over the variable ϕ , equation (28) yields

$$1 = \sum_{n=0}^{\infty} \frac{\pi(n + 2k + m)}{n! \Gamma(n + 2k)} \int_0^{\infty} x^{n+\alpha/2-1/4} I_{\alpha-\frac{1}{2}}^{-1}(2\sqrt{x}) h_0(x) dx |\varphi_n\rangle \langle \varphi_n|. \quad (29)$$

From here it follows that if the equation

$$\frac{\pi(n + 2k + m)}{\Gamma(n + 1)\Gamma(n + 2k)} \int_0^{\infty} x^{n+\alpha/2-1/4} I_{\alpha-\frac{1}{2}}^{-1}(2\sqrt{x}) h_0(x) dx = 1 \quad (30)$$

is satisfied, then (28) will take place also because of the completeness of the system $\{\varphi_n\}$. If now we rewrite (30) as

$$\int_0^{\infty} x^n \Phi(x) dx = \frac{\Gamma(n + 1)\Gamma(n + 2k)}{(n + 2k + p)} \quad (31)$$

where

$$\Phi(x) = |N_{0\lambda}N_{1\lambda}|^2 h(x) \quad (32)$$

we recognize in it a problem of moments on a semiaxis (see, e.g., [32]). To solve this problem, we use the following integral [33]:

$$\int_0^\infty x^n f(x) dx = \Gamma(n + 1)\Gamma(n + 2k) \tag{33}$$

where $f(x) = 2x^{k-\frac{1}{2}}K_{2k-1}(2\sqrt{x})$.

It is not difficult to prove that $\Phi(x)$ can be expressed in terms of $f(x)$ as follows:

$$\Phi(x) = x^{m+2k-1} \int_x^\infty y^{-2k-m} f(y) dy. \tag{34}$$

Indeed, first we note that since $k > \frac{1}{2}$ we have $x\Phi(x) \rightarrow 0$ when $x \rightarrow 0$. Therefore, the integration in (31) by parts under the condition (33) yields just the right-hand side of (31) meaning that equations (34) and (32) define the measure $\tilde{\rho}_1(\lambda)$.

The resolution of identity (28) permits us to construct a holomorphic representation for the transformed Hamiltonian h_1 . The Fourier coefficients $\{c_n\}$ of a function $\varphi(x, t)$ over the basis $\{\varphi_n(x, t)\}$ gives us the same function in CS representation: $\varphi^c(\lambda) = \langle \varphi_\lambda | \varphi \rangle = N\varphi(\lambda)$ where the function $\varphi(\lambda) = \sum_{n=0}^\infty b_n c_n \lambda^n$ is the holomorphic representative of $\varphi(x, t)$. Now one can define a new inner product

$$\langle \varphi_1(\lambda) | \varphi_2(\lambda) \rangle = \int \langle \varphi_1 | \varphi_{\bar{\lambda}} \rangle \langle \varphi_{\bar{\lambda}} | \varphi_2 \rangle d\tilde{\rho}_1(\lambda) = \int |N|^2 \bar{\varphi}_1(\lambda) \varphi_2(\lambda) d\tilde{\rho}_1(\lambda)$$

in the space of holomorphic functions, which gives us a holomorphic representation of states and operators different from that discussed in section 2.2.

One can easily see that the Darboux transformation being realized, any basis function $\psi_n(\lambda)$ goes to $\varphi_n(\lambda) = (n + 2k + m)^{\frac{1}{2}}(2k + m)^{-\frac{1}{2}}\psi_n(\lambda)$. Therefore, if we want the functions $\psi_n(\lambda)$ and $\varphi_n(\lambda)$ to be related by the Darboux transformation, we have to put

$$\varphi_n(\lambda) = (n + 2k + m)^{-\frac{1}{2}}L(\lambda)\psi_n(\lambda) \quad \psi_n(\lambda) = (n + 2k + m)^{-\frac{1}{2}}L^+(\lambda)\varphi_n(\lambda).$$

This gives us the holomorphic representation of the Darboux transformation operators

$$L(\lambda) = (m + 2k)^{-\frac{1}{2}}[k_0(\lambda) - k - m] \quad L^+(\lambda) = (m + 2k)^{\frac{1}{2}}.$$

3.2. Transformation of Perelomov coherent states

Once again we act with the transformation operator L given in (26) but now on $\psi_z(x, t)$ (20) to get the Darboux-transformed Perelomov CS, $\varphi_z(x, t) = N_{1z}L\psi_z(x, t)$. Normalization constant N_{1z} is easily calculated using the equation $\langle L\psi_z | L\psi_z \rangle = \langle \psi_z | L^+L\psi_z \rangle$ and the factorization property of the transformation operators: $N_{1z}^{-2} = m + 2k(1 - |z|^2)$. Their Fourier series in terms of the basis $\{\varphi_n(x, t)\}$ is

$$\varphi_z(x, t) = N_z \sum_{n=0}^\infty b_n z^n \varphi_n(x, t) \quad N_z = N_{0z}N_{1z}(2k + m)^{\frac{1}{2}} \tag{35}$$

$$b_n = a_n(n + 2k + m)^{\frac{1}{2}}(2k + m)^{-\frac{1}{2}}.$$

Here also the series (35) is exactly the same as previously obtained for the time-independent oscillator [5]. We have already found [5] the measure which realizes the resolution of the identity operator in terms of φ_z , constructed a holomorphic representation for the operators and states, found the Kähler potential and symplectic 2-form meaning that we obtained a classical mechanics, which being quantized *à la* Berezin gives us back the holomorphic representation of the quantum system. This procedure can be considered as the one giving rise to a classical counterpart of the Darboux transformation valid for both time-dependent and time-independent cases and consisting of a transformation of the Kähler potential resulting in a distortion of the initial phase space.

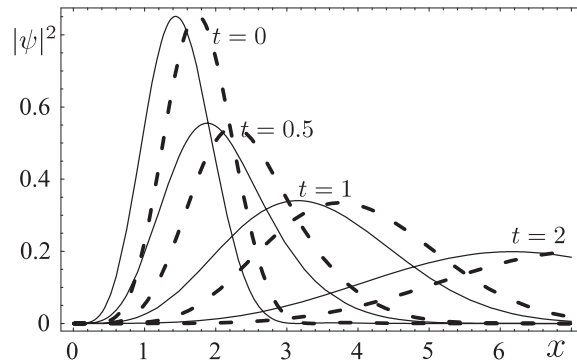


Figure 4. Comparison between probability distribution for Barut–Girardelo CS before (solid line) and after (dashed line) Darboux transformation.

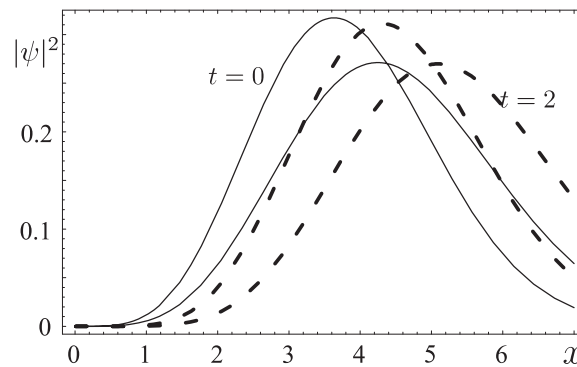


Figure 5. Comparison between probability distribution for Perelomov CS before (solid line) and after (dashed line) Darboux transformation.

3.3. Discussion

We would like to point out that if the Schrödinger equation has a symmetry algebra, this property is usually lost after the Darboux transformation. Nevertheless, if there exist ladder operators for the initial basis functions, the transformed basis may also have them. In such a case, it is possible to look for eigenstates of the annihilation operator and call these functions coherent states. For the case of the time-independent harmonic oscillator potential, this approach was realized in [30]. Since ladder operators now have two derivative orders more with respect to the initial ladder operators, the differential equation they satisfy is rather complicated [31] which makes it difficult to study such states. Our approach has an advantage that the transformation operator (26) is a simple first-order differential operator with the coefficients expressed in terms of elementary functions only. So, it is very easy to operate with it. Moreover, in such a way one can get different systems of CS if they are available for the initial Hamiltonian. Usually different systems of CS exhibit different properties [10, 19]. We conjecture that the Darboux transformation approximately preserves the different behaviour of different CS. To support this conjecture, we plotted the transformed Barut–Girardelo CS together with the transformed Perelomov CS in figure 2. At first sight, the difference between figures 1 and 2 is practically invisible. To show it better, we plotted the Barut–Girardelo CS (solid line) together with their transformed version (dashed line) in figure 4. Figure 5 shows

the Perelomov CS before (solid line) and after (dashed line) the Darboux transformation. It is clearly seen from these figures that for either case the Darboux transformation results mainly in a displacement of the curve while its shape and time behaviour are very little affected.

4. Conclusion

We have shown that acting with the Darboux transformation operator on known CS of the time-dependent singular oscillator gives us states with similar properties. Thus, the ones obtained from Barut–Girardelo CS may be called Barut–Girardelo-like CS while the others, which are produced using Perelomov CS, may be called Perelomov-like CS. Each system of CS admits a resolution of the identity operator which makes it possible to construct different holomorphic representations. A particular example of a free particle in a p state ($\ell = 1$) shows that Barut–Girardelo-like CS are well localized at the initial time moment while Perelomov-like CS are more stable with time evolution. Such behaviour is a reflection of the similar behaviour of corresponding states before the transformation. Therefore, we hope that the new systems of CS may find application in describing similar experiments where the known systems have proven to be useful with the advantage that an additional time-dependent interaction may be included in the Hamiltonian to approximate better experimental results.

Acknowledgments

The work is partially supported by the President Grant of Russia 1743.2003.2, the Spanish Ministerio de Education, Cultura y Deporte Grant SAB2000-0240 and the Spanish MCYT and European FEDER grant BFM2002-03773.

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